CHAPTER
11

NONLINEAR EQUATIONS

58 AUTONOMOUS SYSTEMS. THE PHASE PLANE AND ITS PHENOMENA

There have been two major trends in the historical development of differential equations. The first and oldest is characterized by attempts to find explicit solutions, either in closed form—which is rarely possible—or in terms of power series. In the second, one abandons all hope of solving equations in any traditional sense, and instead concentrates on a search for qualitative information about the general behavior of solutions. We applied this point of view to linear equations in Chapter 4. The qualitative theory of nonlinear equations is totally different. It was founded by Poincaré around 1880, in connection with his work in celestial mechanics, and since that time has been the object of steadily increasing interest on the part of both pure and applied mathematicians.[1]

The theory of linear differential equations has been studied deeply and extensively for the past 200 years, and is a fairly complete and well-rounded body of knowledge. However, very little of a general

[1] See Appendix A for a general account of Poincaré's work in mathematics and science.

nature is known about nonlinear equations. Our purpose in this chapter is to survey some of the central ideas and methods of this subject, and also to demonstrate that it presents a wide variety of interesting and distinctive new phenomena that do not appear in the linear theory. The reader will be surprised to find that most of these phenomena can be treated quite easily without the aid of sophisticated mathematical machinery, and in fact require little more than elementary differential equations and two-dimensional vector algebra.

Why should one be interested in nonlinear differential equations? The basic reason is that many physical systems—and the equations that describe them—are simply nonlinear from the outset. The usual linearizations are approximating devices that are partly confessions of defeat in the face of the original nonlinear problems and partly expressions of the practical view that half a loaf is better than none. It should be added at once that there are many physical situations in which a linear approximation is valuable and adequate for most purposes. This does not alter the fact that in many other situations linearization is unjustified.[2]

It is quite easy to give simple examples of problems that are essentially nonlinear. For instance, if x is the angle of deviation of an undamped pendulum of length a whose bob has mass m, then we saw in Section 5 that its equation of motion is

$$\frac{d^2x}{dt^2} + \frac{g}{a}\sin x = 0; \tag{1}$$

and if there is present a damping force proportional to the velocity of the bob, then the equation becomes

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{g}{a}\sin x = 0. \tag{2}$$

In the usual linearization we replace $\sin x$ by x, which is reasonable for small oscillations but amounts to a gross distortion when x is large. An example of a different type can be found in the theory of the vacuum tube, which leads to the important *van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0. \tag{3}$$

↑ fretn Δt sign

[2] It has even been suggested by Einstein that since the basic equations of physics are nonlinear, all of mathematical physics will have to be done over again. If his crystal ball was clear on the day he said this, the mathematics of the future will certainly be very different from that of the past and present.

will be seen later that each of these nonlinear equations has interesting operties not shared by the others.

Throughout this chapter we shall be concerned with second order nlinear equations of the form

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right), \tag{4}$$

ich includes equations (1), (2), and (3) as special cases. If we imagine imple dynamical system consisting of a particle of unit mass moving on x-axis, and if $f(x, dx/dt)$ is the force acting on it, then (4) is the uation of motion. The values of x (position) and dx/dt (velocity), ich at each instant characterize the state of the system, are called its *ases*, and the plane of the variables x and dx/dt is called the *phase ne*. If we introduce the variable $y = dx/dt$, then (4) can be replaced the equivalent system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = f(x,y). \end{cases} \tag{5}$$

shall see that a good deal can be learned about the solutions of (4) by dying the solutions of (5). When t is regarded as a parameter, then in eral a solution of (5) is a pair of functions $x(t)$ and $y(t)$ defining a ve in the xy-plane, which is simply the phase plane mentioned above. shall be interested in the total picture formed by these curves in the ase plane.

More generally, we study systems of the form

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y), \end{cases} \tag{6}$$

ere F and G are continuous and have continuous first partial ivatives throughout the plane. A system of this kind, in which the ependent variable t does not appear in the functions F and G on the t, is said to be *autonomous*. We now turn to a closer examination of solutions of such a system.

It follows from our assumptions and Theorem 54-A that if t_0 is any nber and (x_0,y_0) is any point in the phase plane, then there exists a que solution

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \tag{7}$$

of (6) such that $x(t_0) = x_0$ and $y(t_0) = y_0$. If $x(t)$ and $y(t)$ are not both constant functions, then (7) defines a curve in the phase plane called a *path* of the system.[3] It is clear that if (7) is a solution of (6), then

$$\begin{cases} x = x(t + c) \\ y = y(t + c) \end{cases} \tag{8}$$

is also a solution for any constant c. Thus each path is represented by many solutions, which differ from one another only by a translation of the parameter. Also, it is quite easy to prove (see Problem 2) that any path through the point (x_0,y_0) must correspond to a solution of the form (8). It follows from this that at most one path passes through each point of the phase plane. Furthermore, the direction of increasing t along a given path is the same for all solutions representing the path. A path is therefore a *directed curve*, and in our figures we shall use arrows to indicate the direction in which the path is traced out as t increases.

The above remarks show that in general the paths of (6) cover the entire phase plane and do not intersect one another. The only exceptions to this statement occur at points (x_0,y_0) where both F and G vanish:

$$F(x_0,y_0) = 0 \qquad \text{and} \qquad G(x_0,y_0) = 0.$$

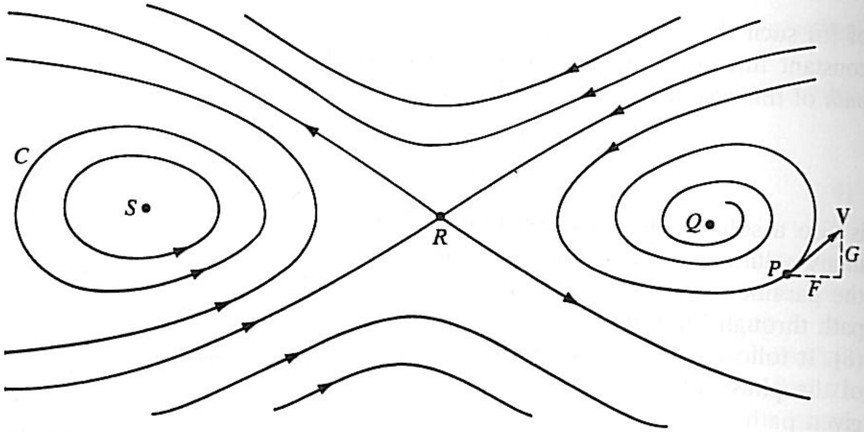
These points are called *critical points*, and at such a point the unique solution guaranteed by Theorem 54-A is the constant solution $x = x_0$ and $y = y_0$. A constant solution does not define a path, and therefore no path goes through a critical point. In our work we will always assume that each critical point (x_0,y_0) is *isolated*, in the sense that there exists a circle centered on (x_0,y_0) that contains no other critical point.

In order to obtain a physical interpretation of critical points, let us consider the special autonomous system (5) arising from the dynamical equation (4). In this case a critical point is a point $(x_0,0)$ at which $y = 0$ and $f(x_0,0) = 0$; that is, it corresponds to a state of the particle's motion in which both the velocity dx/dt and the acceleration $dy/dt = d^2x/dt^2$ vanish. This means that the particle is at rest with no force acting on it, and is therefore in a state of equilibrium.[4] It is obvious that the states of equilibrium of a physical system are among its most important features, and this accounts in part for our interest in critical points.

The general autonomous system (6) does not necessarily arise from any dynamical equation of the form (4). What sort of physical meaning can be attached to the paths and critical points in this case? Here it is convenient to consider Fig. 66 and the two-dimensional vector field

[3] The terms *trajectory* and *characteristic* are used by some writers.

[4] For this reason, some writers use the term *equilibrium point* instead of critical point.



'IGURE 66

lefined by

$$\mathbf{V}(x,y) = F(x,y)\mathbf{i} + G(x,y)\mathbf{j},$$

vhich at a typical point $P = (x,y)$ has horizontal component $F(x,y)$ and ertical component $G(x,y)$. Since $dx/dt = F$ and $dy/dt = G$, this vector ; tangent to the path at P and points in the direction of increasing t. If /e think of t as time, then $\mathbf{V}$ can be interpreted as the velocity vector of a article moving along the path. We can also imagine that the entire phase lane is filled with particles, and that each path is the trail of a moving article preceded and followed by many others on the same path and ccompanied by yet others on nearby paths. This situation can be escribed as a two-dimensional *fluid motion*; and since the system (6) is utonomous, which means that the vector $\mathbf{V}(x,y)$ at a fixed point (x,y) oes not change with time, the fluid motion is *stationary*. The paths are ıe trajectories of the moving particles, and the critical points Q, R, and are points of zero velocity where the particles are at rest (i.e., agnation points of the fluid motion).

The most striking features of the fluid motion illustrated in Fig. 66 'e:

) the critical points;
) the arrangement of the paths near critical points;
) the stability or instability of critical points, that is, whether a particle near such a point remains near or wanders off into another part of the plane;
) closed paths (like C in the figure), which correspond to periodic solutions.

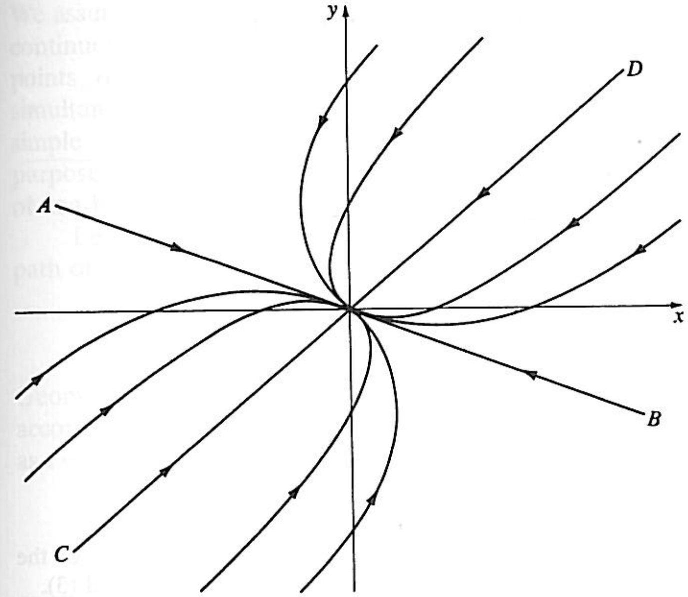


FIGURE 67

These features constitute a major part of the *phase portrait* (or overall picture of the paths) of the system (6). Since in general nonlinear equations and systems cannot be solved explicitly, the purpose of the qualitative theory discussed in this chapter is to discover as much as possible about the phase portrait directly from the functions F and G. To gain some insight into the sort of information we might hope to obtain, observe that if $x(t)$ is a periodic solution of the dynamical equation (4), then its derivative $y(t) = dx/dt$ is also periodic and the corresponding path of the system (5) is therefore closed. Conversely, if any path of (5) is closed, then (4) has a periodic solution. As a concrete example of the application of this idea, we point out that the van der Pol equation—which cannot be solved—can nevertheless be shown to have a unique periodic solution (if $\mu > 0$) by showing that its equivalent autonomous system has a unique closed path. HWK ?

PROBLEMS

1. Derive equation (2) by applying Newton's second law of motion to the bob of the pendulum.
2. Let (x_0,y_0) be a point in the phase plane. If $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$ are solutions of (6) such that $x_1(t_1) = x_0$, $y_1(t_1) = y_0$ and $x_2(t_2) = x_0$, $y_2(t_2) = y_0$ for suitable t_1 and t_2, show that there exists a constant c such that

$$x_1(t + c) = x_2(t) \qquad \text{and} \qquad y_1(t + c) = y_2(t).$$

3. Describe the relation between the phase portraits of the systems

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dx}{dt} = -F(x,y) \\ \dfrac{dy}{dt} = -G(x,y). \end{cases}$$

4. Describe the phase portrait of each of the following systems:

(a) $\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = 0; \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = 2; \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = 0; \end{cases}$ (d) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -y. \end{cases}$

5. The critical points and paths of equation (4) are by definition those of the equivalent system (5). Find the critical points of equations (1), (2), and (3).

6. Find the critical points of

(a) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - (x^3 + x^2 - 2x) = 0;$

(b) $\begin{cases} \dfrac{dx}{dt} = y^2 - 5x + 6 \\ \dfrac{dy}{dt} = x - y. \end{cases}$

7. Find all solutions of the nonautonomous system

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = x + e^t, \end{cases}$$

and sketch (in the xy-plane) some of the curves defined by these solutions.

59 TYPES OF CRITICAL POINTS. STABILITY

Consider an autonomous system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y). \end{cases} \tag{1}$$

We assume, as usual, that the functions F and G are continuous and have continuous first partial derivatives throughout the xy-plane. The critical points of (1) can be found, at least in principle, by solving the simultaneous equations $F(x,y) = 0$ and $G(x,y) = 0$. There are four simple types of critical points that occur quite frequently, and our purpose in this section is to describe them in terms of the configurations of nearby paths. First, however, we need two definitions.

Let (x_0,y_0) be an isolated critical point of (1). If $C = [x(t),y(t)]$ is a path of (1), then we say that C *approaches* (x_0,y_0) as $t \to \infty$ if

$$\lim_{t\to\infty} x(t) = x_0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = y_0.\text{[5]} \tag{2}$$

Geometrically, this means that if $P = (x,y)$ is a point that traces out C in accordance with the equations $x = x(t)$ and $y = y(t)$, then $P \to (x_0,y_0)$ as $t \to \infty$. If it is also true that

$$\lim_{t\to\infty} \frac{y(t) - y_0}{x(t) - x_0} \tag{3}$$

exists, or if the quotient in (3) becomes either positively or negatively infinite as $t \to \infty$, then we say that C *enters the critical point* (x_0,y_0) as $t \to \infty$. The quotient in (3) is the slope of the line joining (x_0,y_0) and the point P with coordinates $x(t)$ and $y(t)$, so the additional requirement means that this line approaches a definite direction as $t \to \infty$. In the above definitions, we may also consider limits as $t \to -\infty$. It is clear that these properties are properties of the path C, and do not depend on which solution is used to represent this path.

It is sometimes possible to find explicit solutions of the system (1), and these solutions can then be used to determine the paths. In most cases, however, to find the paths it is necessary to eliminate t between the two equations of the system, which yields

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}. \tag{4}$$

This first order equation gives the slope of the tangent to the path of (1) that passes through the point (x,y), provided that the functions F and G are not both zero at this point. In this case, of course, the point is a critical point and no path passes through it. The paths of (1) therefore coincide with the one-parameter family of integral curves of (4), and this

[5] It can be proved that if (2) is true for some solution $x(t)$, $y(t)$, then (x_0, y_0) is necessarily a critical point. See F. G. Tricomi, *Differential Equations*, p. 47, Blackie, Glasgow, 1961.

family can often be obtained by the methods of Chapter 2. It should be noted, however, that while the paths of (1) are directed curves, the integral curves of (4) have no direction associated with them. Each of these techniques for determining the paths will be illustrated in the examples below.

We now give geometric descriptions of the four main types of critical points. In each case we assume that the critical point under discussion is the origin $O = (0,0)$.

Nodes. A critical point like that in Fig. 67 is called a *node*. Such a point is approached and also entered by each path as $t \to \infty$ (or as $t \to -\infty$). For the node shown in Fig. 67, there are four half-line paths, AO, BO, CO, and DO, which together with the origin make up the lines AB and CD. All other paths resemble parts of parabolas, and as each of these paths approaches O its slope approaches that of the line AB.

Example 1. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -x + 2y. \end{cases} \tag{5}$$

It is clear that the origin is the only critical point, and the general solution can be found quite easily by the methods of Section 56:

$$\begin{cases} x = c_1 e^t \\ y = c_1 e^t + c_2 e^{2t}. \end{cases} \tag{6}$$

When $c_1 = 0$, we have $x = 0$ and $y = c_2 e^{2t}$. In this case the path (Fig. 68) is the positive y-axis when $c_2 > 0$, and the negative y-axis when $c_2 < 0$, and each path approaches and enters the origin as $t \to -\infty$. When $c_2 = 0$, we have $x = c_1 e^t$ and $y = c_1 e^t$. This path is the half-line $y = x$, $x > 0$, when $c_1 > 0$, and the half-line $y = x$, $x < 0$, when $c_1 < 0$, and again both paths approach and enter the origin as $t \to -\infty$. When both c_1 and c_2 are $\neq 0$, the paths lie on the parabolas $y = x + (c_2/c_1^2)x^2$, which go through the origin with slope 1. It should be understood that each of these paths consists of only part of a parabola, the part with $x > 0$ if $c_1 > 0$, and the part with $x < 0$ if $c_1 < 0$. Each of these paths also approaches and enters the origin as $t \to -\infty$; this can be seen at once from (6). If we proceed directly from (5) to the differential equation

$$\frac{dy}{dx} = \frac{-x + 2y}{x}, \tag{7}$$

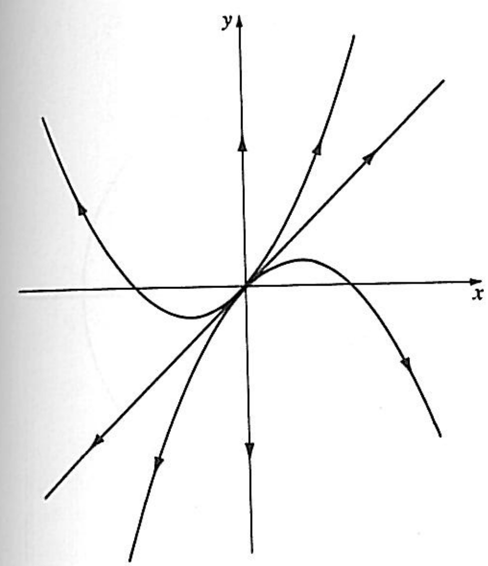


FIGURE 68

giving the slope of the tangent to the path through (x,y) [provided $(x,y) \neq (0,0)$], then on solving (7) as a homogeneous equation, we find that $y = x + cx^2$. This procedure yields the curves on which the paths lie (except those on the y axis), but gives no information about the manner in which the paths are traced out. It is clear from this discussion that the critical point (0,0) of the system (5) is a node.

Saddle points. A critical point like that in Fig. 69 is called a *saddle point*. It is approached and entered by two half-line paths AO and BO as $t \to \infty$, and these two paths lie on a line AB. It is also approached and entered by two half-line paths CO and DO at $t \to -\infty$, and these two paths lie on another line CD. Between the four half-line paths there are four regions, and each contains a family of paths resembling hyperbolas. These paths do not approach O as $t \to \infty$ or as $t \to -\infty$, but instead are asymptotic to one or another of the half-line paths as $t \to \infty$ and as $t \to -\infty$.

Centers. A *center* (sometimes called a *vortex*) is a critical point that is surrounded by a family of closed paths. It is not approached by any path as $t \to \infty$ or as $t \to -\infty$.

Example 2. The system

$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = x \end{cases} \tag{8}$$

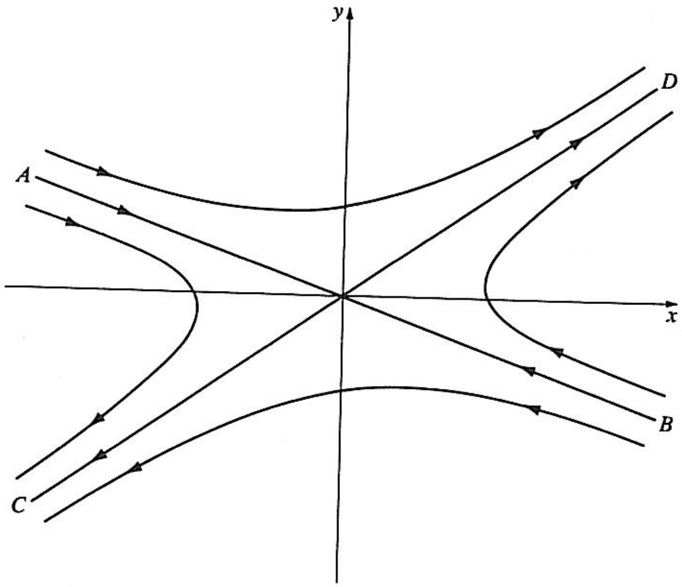


FIGURE 69

has the origin as its only critical point, and its general solution is

$$\begin{cases} x = -c_1 \sin t + c_2 \cos t \\ y = c_1 \cos t + c_2 \sin t. \end{cases} \tag{9}$$

The solution satisfying the conditions $x(0) = 1$ and $y(0) = 0$ is clearly

$$\begin{cases} x = \cos t \\ y = \sin t; \end{cases} \tag{10}$$

and the solution determined by $x(0) = 0$ and $y(0) = -1$ is

$$\begin{cases} x = \sin t = \cos\left(t - \dfrac{\pi}{2}\right) \\ y = -\cos t = \sin\left(t - \dfrac{\pi}{2}\right). \end{cases} \tag{11}$$

These two different solutions define the same path C (Fig. 70), which is evidently the circle $x^2 + y^2 = 1$. Both (10) and (11) show that this path is traced out in the counterclockwise direction. If we eliminate t between the equations of the system, we get

$$\frac{dy}{dx} = -\frac{x}{y},$$

whose general solution $x^2 + y^2 = c^2$ yields all the paths (but without their directions). It is obvious that the critical point (0,0) of the system (8) is a center.

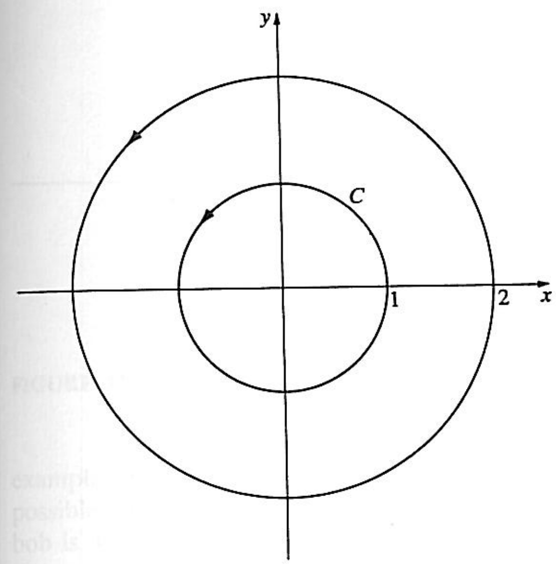


FIGURE 70

Spirals. A critical point like that in Fig. 71 is called a *spiral* (or sometimes a *focus*). Such a point is approached in a spiral-like manner by a family of paths that wind around it an infinite number of times as $t \to \infty$ (or as $t \to -\infty$). Note particularly that while the paths approach O, they do not enter it. That is, a point P moving along such a path approaches O as $t \to \infty$ (or as $t \to -\infty$), but the line OP does not approach any definite direction.

Example 3. If a is an arbitrary constant, then the system

$$\begin{cases} \dfrac{dx}{dt} = ax - y \\ \dfrac{dy}{dt} = x + ay \end{cases} \tag{12}$$

has the origin as its only critical point (why?). The differential equation of the paths,

$$\frac{dy}{dx} = \frac{x + ay}{ax - y}, \tag{13}$$

is most easily solved by introducing polar coordinates r and θ defined by $x = r \cos \theta$ and $y = r \sin \theta$. Since

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \theta = \tan^{-1}\frac{y}{x},$$

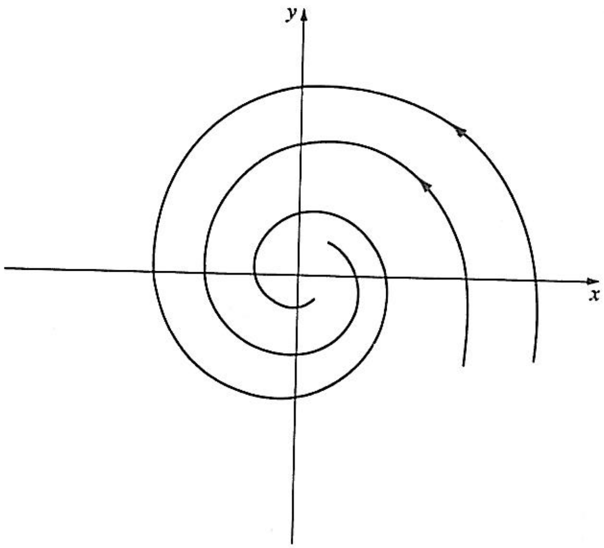


FIGURE 71

we see that

$$r\frac{dr}{dx} = x + y\frac{dy}{dx} \qquad \text{and} \qquad r^2\frac{d\theta}{dx} = x\frac{dy}{dx} - y.$$

With the aid of these equations, (13) can easily be written in the very simple form

$$\frac{dr}{d\theta} = ar,$$

so

$$r = ce^{a\theta} \tag{14}$$

is the polar equation of the paths. The two possible spiral configurations are shown in Fig. 72 and the direction in which these paths are traversed can be seen from the fact that $dx/dt = -y$ when $x = 0$. If $a = 0$, then (12) collapses to (8) and (14) becomes $r = c$, which is the polar equation of the family $x^2 + y^2 = c^2$ of all circles centered on the origin. This example therefore generalizes Example 2; and since the center shown in Fig. 70 stands on the borderline between the spirals of Fig. 72, a critical point that is a center is often called a *borderline case*. We will encounter other borderline cases in the next section.

We now introduce the concept of *stability* as it applies to the critical oints of the system (1).

It was pointed out in the previous section that one of the most portant questions in the study of a physical system is that of its steady ates. However, a steady state has little physical significance unless it has reasonable degree of permanence, i.e., unless it is stable. As a simple

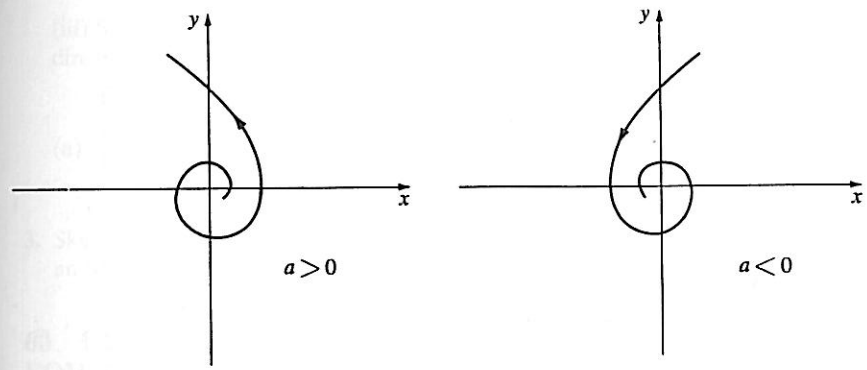


FIGURE 72

example, consider the pendulum of Fig. 73. There are two steady states possible here: when the bob is at rest at the highest point, and when the bob is at rest at the lowest point. The first state is clearly unstable, and the second is stable. We now recall that a steady state of a simple physical system corresponds to an equilibrium point (or critical point) in the phase plane. These considerations suggest in a general way that a small disturbance at an unstable equilibrium point leads to a larger and larger departure from this point, while the opposite is true at a stable equilibrium point.

We now formulate these intuitive ideas in a more precise way. Consider an isolated critical point of the system (1), and assume for the sake of convenience that this point is located at the origin $O = (0,0)$ of the phase plane. This critical point is said to be *stable* if for each positive number R there exists a positive number $r \leq R$ such that every path which is inside the circle $x^2 + y^2 = r^2$ for some $t = t_0$ remains inside the

FIGURE 73

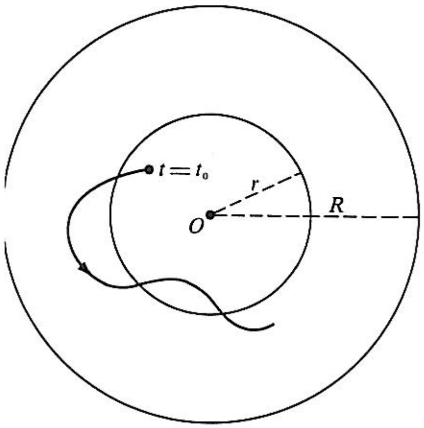


FIGURE 74

ircle $x^2 + y^2 = R^2$ for all $t > t_0$ (Fig. 74). Loosely speaking, a critical oint is stable if all paths that get sufficiently close to the point stay close o the point. Further, our critical point is said to be *asymptotically stable* f it is stable and there exists a circle $x^2 + y^2 = r_0^2$ such that every path vhich is inside this circle for some $t = t_0$ approaches the origin as $t \to \infty$. inally, if our critical point is not stable, then it is called *unstable*.

As examples of these concepts, we point out that the node in Fig. 8, the saddle point in Fig. 69, and the spiral on the left in Fig. 72 are nstable, while the center in Fig. 70 is stable but not asymptotically table. The node in Fig. 67, the spiral in Fig. 71, and the spiral on the ght in Fig. 72 are asymptotically stable.

ROBLEMS

For each of the following nonlinear systems: (i) find the critical points; (ii) find the differential equation of the paths; (iii) solve this equation to find the paths; and (iv) sketch a few of the paths and show the direction of increasing t.

(a) $\begin{cases} \dfrac{dx}{dt} = y(x^2+1) \\ \dfrac{dy}{dt} = 2xy^2; \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = e^y \\ \dfrac{dy}{dt} = e^y \cos x; \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = y(x^2+1) \\ \dfrac{dy}{dt} = -x(x^2+1); \end{cases}$ (d) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = 2x^2y^2. \end{cases}$

Each of the following linear systems has the origin as an isolated critical point. (i) Find the general solution. (ii) Find the differential equation of the paths. (iii) Solve the equation found in (ii) and sketch a few of the paths, showing the direction of increasing t. (iv) Discuss the stability of the critical point.

(a) $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y; \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -2y; \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = 4y \\ \dfrac{dy}{dt} = -x. \end{cases}$

3. Sketch the phase portrait of the equation $d^2x/dt^2 = 2x^3$, and show that it has an unstable isolated critical point at the origin.

60 CRITICAL POINTS AND STABILITY FOR LINEAR SYSTEMS

Our goal in this chapter is to learn as much as we can about nonlinear differential equations by studying the phase portraits of nonlinear autonomous systems of the form

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y). \end{cases}$$

One aspect of this is the problem of classifying the critical points of such a system with respect to their nature and stability. It will be seen in Section 62 that under suitable conditions this problem can be solved for a given nonlinear system by studying a related linear system. We therefore devote this section to a complete analysis of the critical points of linear autonomous systems.

We consider the system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y, \end{cases} \tag{1}$$

which has the origin (0,0) as an obvious critical point. We assume throughout this section that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \tag{2}$$

so that (0,0) is the only critical point. It was proved in Section 56 that (1) has a nontrivial solution of the form

$$\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$$

whenever m is a root of the quadratic equation

$$m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0, \tag{3}$$

which is called the *auxiliary equation* of the system. Observe that condition (2) implies that zero cannot be a root of (3).

Let m_1 and m_2 be the roots of (3). We shall prove that the nature of the critical point (0,0) of the system (1) is determined by the nature of the numbers m_1 and m_2. It is reasonable to expect that three possibilities will occur, according as m_1 and m_2 are real and distinct, real and equal, or conjugate complex. Unfortunately the situation is a little more complicated than this, and it is necessary to consider five cases, subdivided as follows.

Major cases:

Case A. The roots m_1 and m_2 are real, distinct, and of the same sign (node).

Case B. The roots m_1 and m_2 are real, distinct, and of opposite signs (saddle point).

Case C. The roots m_1 and m_2 are conjugate complex but not pure imaginary (spiral).

Borderline cases:

Case D. The roots m_1 and m_2 are real and equal (node).

Case E. The roots m_1 and m_2 are pure imaginary (center).

The reason for the distinction between the major cases and the borderline cases will become clear in Section 62. For the present it suffices to remark that while the borderline cases are of mathematical interest they have little significance for applications, because the circumstances defining them are unlikely to arise in physical problems. We now turn to the proofs of the assertions in parentheses.

Case A. If the roots m_1 and m_2 are real, distinct, and of the same sign, then the critical point (0,0) is a node.

Proof. We begin by assuming that m_1 and m_2 are both negative, and we choose the notation so that $m_1 < m_2 < 0$. By Section 56, the general solution of (1) in this case is

$$\begin{cases} x = c_1A_1e^{m_1t} + c_2A_2e^{m_2t} \\ y = c_1B_1e^{m_1t} + c_2B_2e^{m_2t}, \end{cases} \tag{4}$$

where the A's and B's are definite constants such that $B_1/A_1 \neq B_2/A_2$, and where the c's are arbitrary constants. When $c_2 = 0$, we obtain the solutions

$$\begin{cases} x = c_1A_1e^{m_1t} \\ y = c_1B_1e^{m_1t}, \end{cases} \tag{5}$$

and when $c_1 = 0$, we obtain the solutions

$$\begin{cases} x = c_2A_2e^{m_2t} \\ y = c_2B_2e^{m_2t}. \end{cases} \tag{6}$$

For any $c_1 > 0$, the solution (5) represents a path consisting of half of the line $A_1y = B_1x$ with slope B_1/A_1; and for any $c_1 < 0$, it represents a path consisting of the other half of this line (the half on the other side of the origin). Since $m_1 < 0$, both of these half-line paths approach (0,0) as $t \to \infty$; and since $y/x = B_1/A_1$, both enter (0,0) with slope B_1/A_1 (Fig. 75). In exactly the same way, the solutions (6) represent two half-line paths lying on the line $A_2y = B_2x$ with slope B_2/A_2. These two paths also approach (0,0) as $t \to \infty$, and enter it with slope B_2/A_2.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (4) represents curved paths. Since $m_1 < 0$ and $m_2 < 0$, these paths also approach (0,0) as $t \to \infty$. Furthermore, since $m_1 - m_2 < 0$ and

$$\frac{y}{x} = \frac{c_1B_1e^{m_1t} + c_2B_2e^{m_2t}}{c_1A_1e^{m_1t} + c_2A_2e^{m_2t}} = \frac{(c_1B_1/c_2)e^{(m_1-m_2)t} + B_2}{(c_1A_1/c_2)e^{(m_1-m_2)t} + A_2},$$

it is clear that $y/x \to B_2/A_2$ as $t \to \infty$, so all of these paths enter (0,0) with slope B_2/A_2. Figure 75 presents a qualitative picture of the situation. It is evident that our critical point is a node, and that it is asymptotically stable.

If m_1 and m_2 are both positive, and if we choose the notation so that $m_1 > m_2 > 0$, then the situation is exactly the same except that all the paths now approach and enter (0,0) as $t \to -\infty$. The picture of the paths

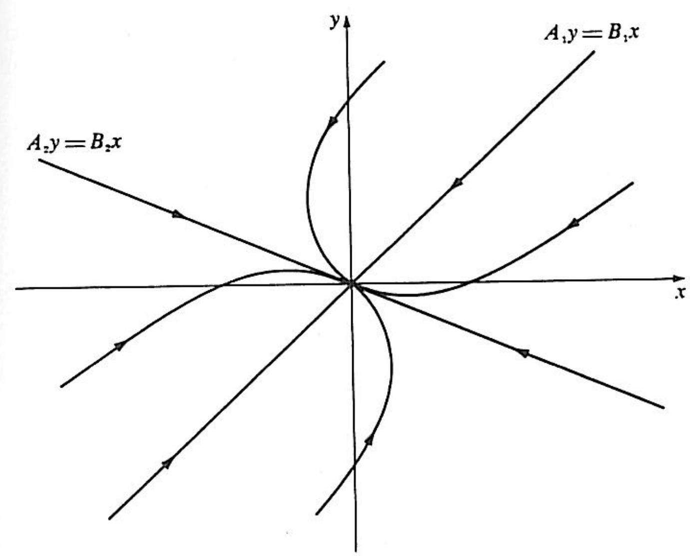


FIGURE 75

given in Fig. 75 is unchanged except that the arrows showing their directions are all reversed. We still have a node, but now it is unstable.

Case B. If the roots m_1 and m_2 are real, distinct, and of opposite signs, then the critical point (0,0) is a saddle point.

***Proof*.** We may choose the notation so that $m_1 < 0 < m_2$. The general solution of (1) can still be written in the form (4), and again we have particular solutions of the forms (5) and (6). The two half-line paths represented by (5) still approach and enter (0,0) as $t \to \infty$, but this time the two half-line paths represented by (6) approach and enter (0,0) as $t \to -\infty$. If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (4) still represents curved paths, but since $m_1 < 0 < m_2$, none of these paths approaches (0,0) as $t \to \infty$ or $t \to -\infty$. Instead, as $t \to \infty$, each of these paths is asymptotic to one of the half-line paths represented by (6); and as $t \to -\infty$, each is asymptotic to one of the half-line paths represented by (5). Figure 76 gives a qualitative picture of this behavior. In this case the critical point is a saddle point, and it is obviously unstable.

Case C. If the roots m_1 and m_2 are conjugate complex but not pure imaginary, then the critical point (0,0) is a spiral.

***Proof*.** In this case we can write m_1 and m_2 in the form $a \pm ib$ where a and b are nonzero real numbers. Also, for later use, we observe that the

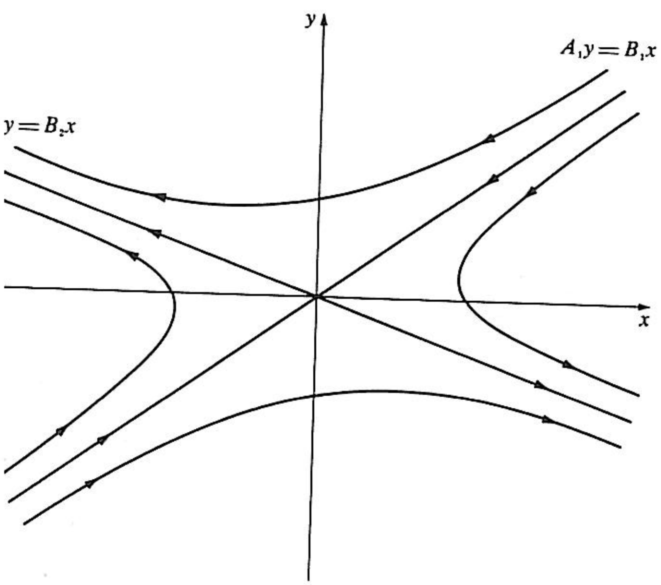


FIGURE 76

discriminant D of equation (3) is negative:

$$\begin{aligned} D &= (a_1 + b_2)^2 - 4(a_1 b_2 - a_2 b_1) \\ &= (a_1 - b_2)^2 + 4a_2 b_1 < 0. \end{aligned} \tag{7}$$

By Section 56, the general solution of (1) in this case is

$$\begin{cases} x = e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)] \\ y = e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)], \end{cases} \tag{8}$$

where the A's and B's are definite constants and the c's are arbitrary constants.

Let us first assume that $a < 0$. Then it is clear from formulas (8) that $x \to 0$ and $y \to 0$ as $t \to \infty$, so all the paths approach (0,0) as $t \to \infty$. We now prove that the paths do not enter the point (0,0) as $t \to \infty$, but instead wind around it in a spiral-like manner. To accomplish this we introduce the polar cordinate θ and show that, along any path, $d\theta/dt$ is either positive for all t or negative for all t. We begin with the fact that $\theta = \tan^{-1}(y/x)$, so

$$\frac{d\theta}{dt} = \frac{x\,dy/dt - y\,dx/dt}{x^2 + y^2};$$

and by using equations (1) we obtain

$$\frac{d\theta}{dt} = \frac{a_2 x^2 + (b_2 - a_1)xy - b_1 y^2}{x^2 + y^2}. \tag{9}$$

Since we are interested only in solutions that represent paths, we assume that $x^2 + y^2 \neq 0$. Now (7) implies that a_2 and b_1 have opposite signs. We consider the case in which $a_2 > 0$ and $b_1 < 0$. When $y = 0$, (9) yields $d\theta/dt = a_2 > 0$. If $y \neq 0$, $d\theta/dt$ cannot be 0; for if it were, then (9) would imply that

$$a_2 x^2 + (b_2 - a_1)xy - b_1 y^2 = 0$$

or

$$a_2\left(\frac{x}{y}\right)^2 + (b_2 - a_1)\frac{x}{y} - b_1 = 0 \tag{10}$$

for some real number x/y—and this cannot be true because the discriminant of the quadratic equation (10) is D, which is negative by (7). This shows that $d\theta/dt$ is always positive when $a_2 > 0$, and in the same way we see that it is always negative when $a_2 < 0$. Since by (8), x and y change sign infinitely often as $t \to \infty$, all paths must spiral in to the origin (counterclockwise or clockwise according as $a_2 > 0$ or $a_2 < 0$). The critical point in this case is therefore a spiral, and it is asymptotically stable.

If $a > 0$, the situation is the same except that the paths approach (0,0) as $t \to -\infty$ and the critical point is unstable. Figure 72 illustrates the arrangement of the paths when $a_2 > 0$.

Case D. If the roots m_1 and m_2 are real and equal, then the critical point (0,0) is a node.

Proof. We begin by assuming that $m_1 = m_2 = m < 0$. There are two subcases that require separate discussion: (i) $a_1 = b_2 \neq 0$ and $a_2 = b_1 = 0$; (ii) all other possibilities leading to a double root of equation (3).

We first consider the subcase (i), which is the situation described in the footnote in Section 56. If a denotes the common value of a_1 and b_2, then equation (3) becomes $m^2 - 2am + a^2 = 0$ and $m = a$. The system (1) is thus

$$\begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dy}{dt} = ay, \end{cases}$$

and its general solution is

$$\begin{cases} x = c_1 e^{mt} \\ y = c_2 e^{mt}, \end{cases} \tag{11}$$

where c_1 and c_2 are arbitrary constants. The paths defined by (11) are half-lines of all possible slopes (Fig. 77), and since $m < 0$ we see that each path approaches and enters (0,0) as $t \to \infty$. The critical point is therefore a node, and it is asymptotically stable. If $m > 0$, we have the same situation except that the paths enter (0,0) as $t \to -\infty$, the arrows in Fig. 77 are reversed, and (0,0) is unstable.

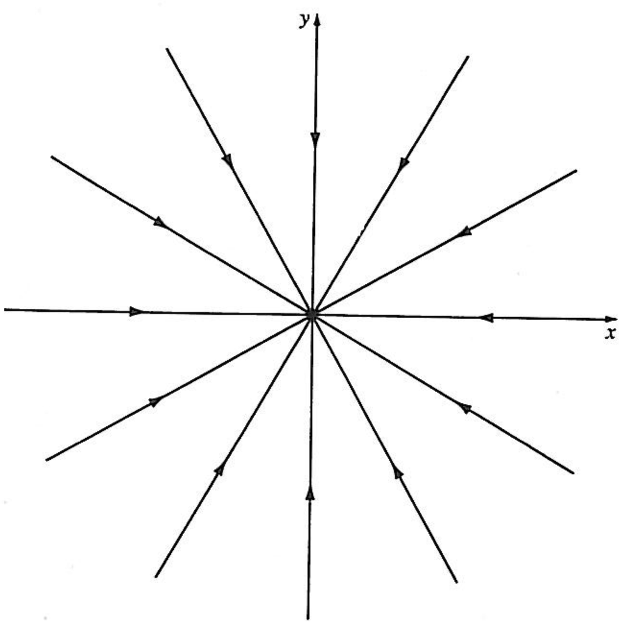


FIGURE 77

We now discuss subcase (ii). By formulas 56-(20) and Problem 56-(4), the general solution of (1) can be written in the form

$$\begin{cases} x = c_1 A e^{mt} + c_2(A_1 + At)e^{mt} \\ y = c_1 B e^{mt} + c_2(B_1 + Bt)e^{mt}, \end{cases} \tag{12}$$

where the A's and B's are definite constants and the c's are arbitrary constants. When $c_2 = 0$, we obtain the solutions

$$\begin{cases} x = c_1 A e^{mt} \\ y = c_1 B e^{mt}. \end{cases} \tag{13}$$

We know that these solutions represent two half-line paths lying on the line $Ay = Bx$ with slope B/A, and since $m < 0$ both paths approach (0,0) as $t \to \infty$ (Fig. 78). Also, since $y/x = B/A$, both paths enter (0,0) with slope B/A. If $c_2 \neq 0$, the solutions (12) represent curved paths, and since $m < 0$ it is clear from (12) that these paths approach (0,0) as $t \to \infty$. Furthermore, it follows from

$$\frac{y}{x} = \frac{c_1 B e^{mt} + c_2(B_1 + Bt)e^{mt}}{c_1 A e^{mt} + c_2(A_1 + At)e^{mt}} = \frac{c_1 B/c_2 + B_1 + Bt}{c_1 A/c_2 + A_1 + At}$$

that $y/x \to B/A$ as $t \to \infty$, so these curved paths all enter (0,0) with slope B/A. We also observe that $y/x \to B/A$ as $t \to -\infty$. Figure 78 gives a qualitative picture of the arrangement of these paths. It is clear that (0,0) is a node that is asymptotically stable. If $m > 0$, the situation is unchanged

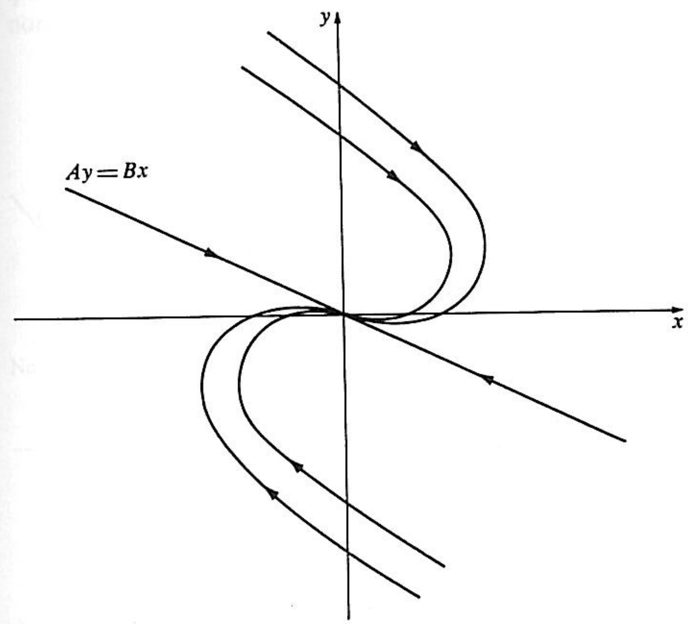


FIGURE 78

except that the directions of the paths are reversed and the critical point is unstable.

Case E. If the roots m_1 and m_2 are pure imaginary, then the critical point (0,0) is a center.

Proof. It suffices here to refer back to the discussion of Case C, for now m_1 and m_2 are of the form $a \pm ib$ with $a = 0$ and $b \neq 0$. The general solution of (1) is therefore given by (8) with the exponential factor missing, so $x(t)$ and $y(t)$ are periodic and each path is a closed curve surrounding the origin. As Fig. 79 suggests, these curves are actually ellipses; this can be proved (see Problem 5) by solving the differential equation of the paths,

$$\frac{dy}{dx} = \frac{a_2x + b_2y}{a_1x + b_1y}. \tag{14}$$

Our critical point (0,0) is evidently a center that is stable but not asymptotically stable.

In the above discussions we have made a number of statements about stability. It will be convenient to summarize this information as follows.

Theorem A. *The critical point* (0,0) *of the linear system* (1) *is stable if and only if both roots of the auxiliary equation* (3) *have nonpositive real parts, and it is asymptotically stable if and only if both roots have negative real parts.*

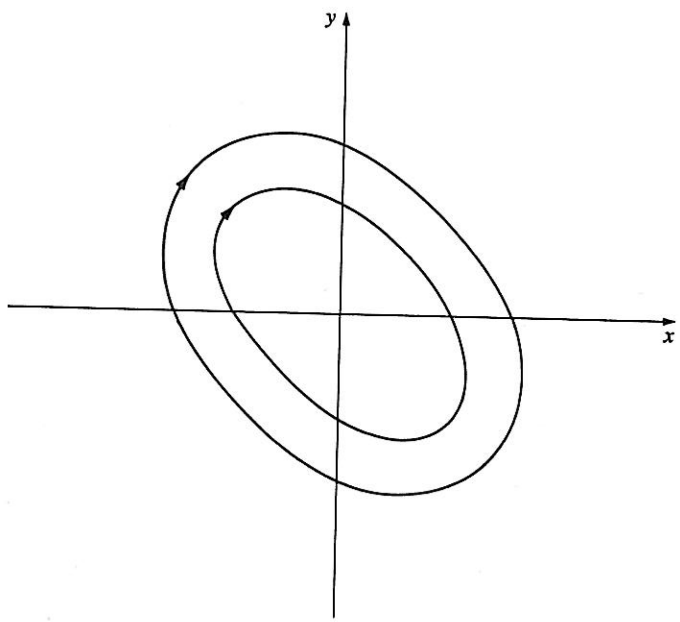


FIGURE 79

If we now write equation (3) in the form

$$(m - m_1)(m - m_2) = m^2 + pm + q = 0, \tag{15}$$

so that $p = -(m_1 + m_2)$ and $q = m_1m_2$, then our five cases can be described just as readily in terms of the coefficients p and q as in terms of the roots m_1 and m_2. In fact, if we interpret these cases in the pq-plane, then we arrive at a striking diagram (Fig. 80) that displays at a glance the nature and stability properties of the critical point (0,0). The first thing to notice is that the p-axis $q = 0$ is excluded, since by condition (2) we know that $m_1m_2 \neq 0$. In the light of what we have learned about our five cases, all of the information contained in the diagram follows directly from the fact that

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Thus, above the parabola $p^2 - 4q = 0$, we have $p^2 - 4q < 0$, so m_1 and m_2 are conjugate complex numbers that are pure imaginary if and only if $p = 0$; these are Cases C and E comprising the spirals and centers. Below the p-axis we have $q < 0$, which means that m_1 and m_2 are real, distinct, and have opposite signs; this yields the saddle points of Case B. And finally, the zone between these two regions (including the parabola but excluding the p-axis) is characterized by the relations $p^2 - 4q \geq 0$ and $q > 0$, so m_1 and m_2 are real and of the same sign; here we have the nodes of Cases A and D. Furthermore, it is clear that there is precisely

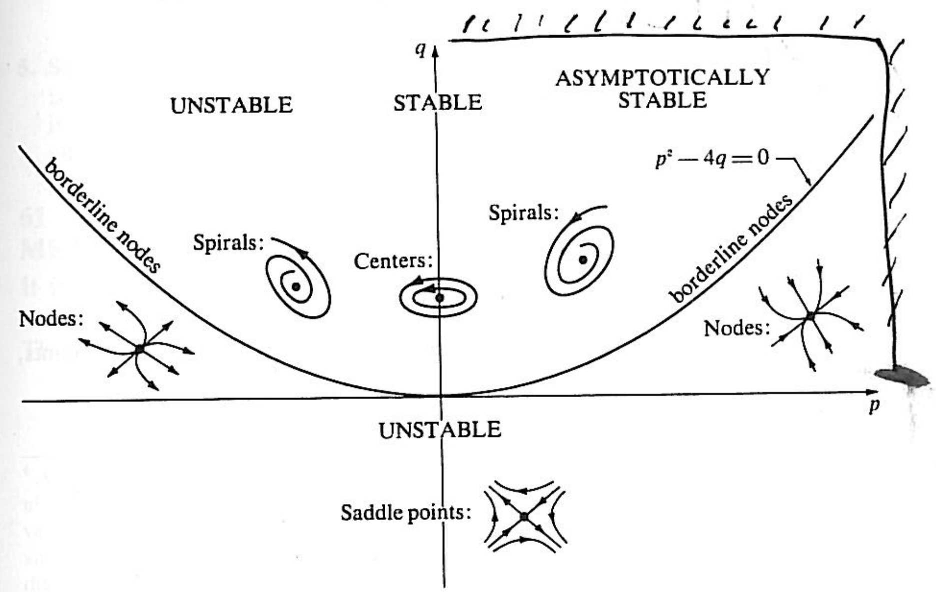


FIGURE 80

one region of asymptotic stability: the first quadrant. We state this formally as follows.

Theorem B. *The critical point (0,0) of the linear system (1) is asymptotically stable if and only if the coefficients* $p = -(a_1 + b_2)$ *and* $q = a_1b_2 - a_2b_1$ *of the auxiliary equation (3) are both positive.*

Finally, it should be emphasized that we have studied the paths of our linear system near a critical point by analyzing explicit solutions of the system. In the next two sections we enter more fully into the spirit of the subject by investigating similar problems for nonlinear systems, which in general cannot be solved explicitly.

PROBLEMS

1. Determine the nature and stability properties of the critical point (0,0) for each of the following linear autonomous systems:

(a) $\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = 3y; \end{cases}$ (e) $\begin{cases} \dfrac{dx}{dt} = -4x - y \\ \dfrac{dy}{dt} = x - 2y; \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = -x - 2y \\ \dfrac{dy}{dt} = 4x - 5y; \end{cases}$ (f) $\begin{cases} \dfrac{dx}{dt} = 4x - 3y \\ \dfrac{dy}{dt} = 8x - 6y; \end{cases}$

(c) $\begin{cases} \dfrac{dx}{dt} = -3x + 4y \\ \dfrac{dy}{dt} = -2x + 3y; \end{cases}$ (g) $\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = 5x + 2y. \end{cases}$

(d) $\begin{cases} \dfrac{dx}{dt} = 5x + 2y \\ \dfrac{dy}{dt} = -17x - 5y; \end{cases}$

2. If $a_1b_2 - a_2b_1 = 0$, show that the system (1) has infinitely many critical points, none of which are isolated.

3. (a) If $a_1b_2 - a_2b_1 \neq 0$, show that the system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + c_1 \\ \dfrac{dy}{dt} = a_2x + b_2y + c_2 \end{cases}$$

has a single isolated critical point (x_0, y_0).

(b) Show that the system in (a) can be written in the form of (1) by means of the change of variables $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$.

(c) Find the critical point of the system

$$\begin{cases} \dfrac{dx}{dt} = 2x - 2y + 10 \\ \dfrac{dy}{dt} = 11x - 8y + 49, \end{cases}$$

write the system in the form of (1) by changing the variables, and determine the nature and stability properties of the critical point.

4. In Section 20 we studied the free vibrations of a mass attached to a spring by solving the equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + a^2x = 0,$$

where $b \geq 0$ and $a > 0$ are constants representing the viscosity of the medium and the stiffness of the spring, respectively. Consider the equivalent autonomous system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -a^2x - 2by, \end{cases} \quad (*)$$

which has (0,0) as its only critical point.

(a) Find the auxiliary equation of (*). What are p and q?

(b) For each of the following four cases, describe the nature and stability properties of the critical point, and give a brief physical interpretation of the corresponding motion of the mass:

(i) $b = 0$; (iii) $b = a$;
(ii) $0 < b < a$; (iv) $b > a$.

5. Solve equation (14) under the hypotheses of Case E, and show that the result is a one-parameter family of ellipses surrounding the origin. *Hint*: Recall that if $Ax^2 + Bxy + Cy^2 = D$ is the equation of a real curve, then the curve is an ellipse if and only if the discriminant $B^2 - 4AC$ is negative.

61 STABILITY BY LIAPUNOV'S DIRECT METHOD

It is intuitively clear that if the total energy of a physical system has a local minimum at a certain equilibrium point, then that point is stable. This idea was generalized by Liapunov[6] into a simple but powerful

[6] Alexander Mikhailovich Liapunov (1857–1918) was a Russian mathematician and mechanical engineer. He had the very rare merit of producing a doctoral dissertation of lasting value. This classic work was originally published in 1892 in Russian, but is now available in an English translation, *Stability of Motion*, Academic Press, New York, 1966. Liapunov died by violence in Odessa, which cannot be considered a surprising fate for a middle-class intellectual in the chaotic aftermath of the Russian Revolution.

method for studying stability problems in a broader context. We shall discuss Liapunov's method and some of its applications in this and the next section.

Consider an autonomous system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y), \end{cases} \tag{1}$$

and assume that this system has an isolated critical point, which as usual we take to be the origin (0,0).[7] Let $C = [x(t), y(t)]$ be a path of (1), and consider a function $E(x,y)$ that is continuous and has continuous first partial derivatives in a region containing this path. If a point (x,y) moves along the path in accordance with the equations $x = x(t)$ and $y = y(t)$, then $E(x,y)$ can be regarded as a function of t along C [we denote this function by $E(t)$] and its rate of change is

$$\begin{aligned} \frac{dE}{dt} &= \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G. \end{aligned} \tag{2}$$

This formula is at the heart of Liapunov's ideas, and in order to exploit it we need several definitions that specify the kinds of functions we shall be interested in.

Suppose that $E(x,y)$ is continuous and has continuous first partial derivatives in some region containing the origin. If E vanishes at the origin, so that $E(0,0) = 0$, then it is said to be *positive definite* if $E(x,y) > 0$ for $(x,y) \neq (0,0)$, and *negative definite* if $E(x,y) < 0$ for $(x,y) \neq (0,0)$. Similarly, E is called *positive semidefinite* if $E(0,0) = 0$ and $E(x,y) \geq 0$ for $(x,y) \neq (0,0)$, and *negative semidefinite* if $E(0,0) = 0$ and $E(x,y) \leq 0$ for $(x,y) \neq (0,0)$. It is clear that functions of the form $ax^{2m} + by^{2n}$, where a and b are positive constants and m and n are positive integers, are positive definite. Since $E(x,y)$ is negative definite if and only if $-E(x,y)$ is positive definite, functions of the form $ax^{2m} + by^{2n}$ with $a < 0$ and $b < 0$ are negative definite. The functions x^{2m}, y^{2m}, and $(x - y)^{2m}$ are not positive definite, but are nevertheless positive

[7] A critical point (x_0, y_0) can always be moved to the origin by a simple translation of coordinates $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$, so there is no loss of generality in assuming that it lies at the origin in the first place.

semidefinite. If $E(x,y)$ is positive definite, then $z = E(x,y)$ can be interpreted as the equation of a surface (Fig. 81) that resembles a paraboloid opening upward and tangent to the xy-plane at the origin.

A positive definite function $E(x,y)$ with the property that

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G \tag{3}$$

is negative semidefinite is called a *Liapunov function* for the system (1). By formula (2), the requirement that (3) be negative semidefinite means that $dE/dt \leq 0$—and therefore E is nonincreasing—along the paths of (1) near the origin. These functions generalize the concept of the total energy of a physical system. Their relevance for stability problems is made clear in the following theorem, which is Liapunov's basic discovery.

Theorem A. *If there exists a Liapunov function $E(x,y)$ for the system* (1), *then the critical point* (0,0) *is stable. Furthermore, if this function has the additional property that the function* (3) *is negative definite, then the critical point* (0,0) *is asymptotically stable.*

Proof. Let C_1 be a circle of radius $R > 0$ centered on the origin (Fig. 82), and assume also that C_1 is small enough to lie entirely in the domain of definition of the function E. Since $E(x,y)$ is continuous and positive

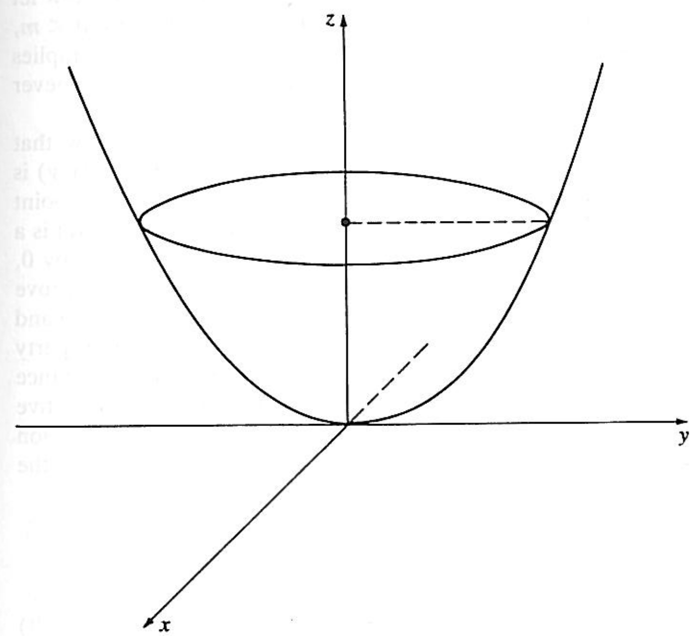


FIGURE 81

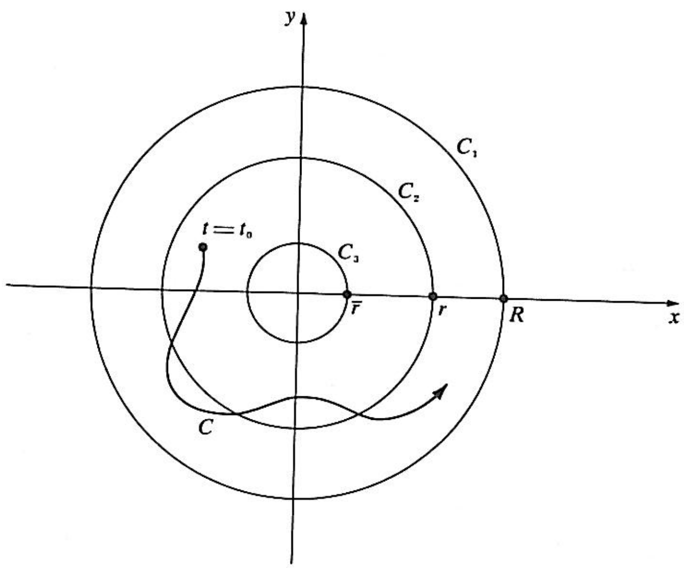


FIGURE 82

definite, it has a positive minimum m on C_1. Next, $E(x,y)$ is continuous at the origin and vanishes there, so we can find a positive number $r < R$ such that $E(x,y) < m$ whenever (x,y) is inside the circle C_2 of radius r. Now let $C = [x(t), y(t)]$ be any path which is inside C_2 for $t = t_0$. Then $E(t_0) < m$, and since (3) is negative semidefinite we have $dE/dt \leq 0$, which implies that $E(t) \leq E(t_0) < m$ for all $t > t_0$. It follows that the path C can never reach the circle C_1 for any $t > t_0$, so we have stability.

To prove the second part of the theorem, it suffices to show that under the additional hypothesis we also have $E(t) \to 0$, for since $E(x,y)$ is positive definite this will imply that the path C approaches the critical point (0,0). We begin by observing that since $dE/dt < 0$, it follows that $E(t)$ is a decreasing function; and since by hypothesis $E(t)$ is bounded below by 0, we conclude that $E(t)$ approaches some limit $L \geq 0$ as $t \to \infty$. To prove that $E(t) \to 0$ it suffices to show that $L = 0$, so we assume that $L > 0$ and deduce a contradiction. Choose a positive number $\bar{r} < r$ with the property that $E(x,y) < L$ whenever (x,y) is inside the circle C_3 with radius $\bar{r}$. Since the function (3) is continuous and negative definite, it has a negative maximum $-k$ in the ring consisting of the circles C_1 and C_3 and the region between them. This ring contains the entire path C for $t \geq t_0$, so the equation

$$E(t) = E(t_0) + \int_{t_0}^{t} \frac{dE}{dt}\,dt$$

yields the inequality

$$E(t) \leq E(t_0) - k(t - t_0) \tag{4}$$

for all $t \geq t_0$. However, the right side of (4) becomes negatively infinite as $t \to \infty$, so $E(t) \to -\infty$ as $t \to \infty$. This contradicts the fact that $E(x,y) \geq 0$, so we conclude that $L = 0$ and the proof is complete.

Example 1. Consider the equation of motion of a mass m attached to a spring:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0. \tag{5}$$

Here $c \geq 0$ is a constant representing the viscosity of the medium through which the mass moves, and $k > 0$ is the spring constant. The autonomous system equivalent to (5) is

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{k}{m}x - \dfrac{c}{m}y, \end{cases} \tag{6}$$

and its only critical point is (0,0). The kinetic energy of the mass is $my^2/2$, and the potential energy (or the energy stored in the spring) is

$$\int_0^x kx\,dx = \frac{1}{2}kx^2.$$

Thus the total energy of the system is

$$E(x,y) = \frac{1}{2}my^2 + \frac{1}{2}kx^2. \tag{7}$$

It is easy to see that (7) is positive definite; and since

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G = kxy + my\left(-\frac{k}{m}x - \frac{c}{m}y\right)$$
$$= -cy^2 \leq 0,$$

(7) is a Liapunov function for (6) and the critical point (0,0) is stable. We know from Problem 60-4 that when $c > 0$ this critical point is asymptotically stable, but the particular Liapunov function discussed here is not capable of detecting this fact.[8]

[8] It is known that both stability and asymptotic stability can always be detected by suitable Liapunov functions, but knowing in principle that such a function exists is a very different matter from actually finding one. For references on this point, see L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, p. 111, Academic Press, New York, 1963; or G. Sansone and R. Conti, *Non-Linear Differential Equations*, p. 481, Macmillan, New York, 1964.

Example 2. The system

$$\begin{cases} \dfrac{dx}{dt} = -2xy \\ \dfrac{dy}{dt} = x^2 - y^3 \end{cases} \tag{8}$$

has (0,0) as an isolated critical point. Let us try to prove stability by constructing a Liapunov function of the form $E(x,y) = ax^{2m} + by^{2n}$. It is clear that

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G = 2max^{2m-1}(-2xy) + 2nby^{2n-1}(x^2 - y^3)$$
$$= (-4max^{2m}y + 2nbx^2y^{2n-1}) - 2nby^{2n+2}.$$

We wish to make the expression in parentheses vanish, and inspection shows that this can be done by choosing $m = 1$, $n = 1$, $a = 1$, and $b = 2$. With these choices we have $E(x,y) = x^2 + 2y^2$ (which is positive definite) and $(\partial E/\partial x)F + (\partial E/\partial y)G = -4y^4$ (which is negative semidefinite). The critical point (0,0) of the system (8) is therefore stable.

It is clear from this example that in complicated situations it may be very difficult indeed to construct suitable Liapunov functions. The following result is sometimes helpful in this connection.

Theorem B *The function $E(x,y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$, and is negative definite if and only if $a < 0$ and $b^2 - 4ac < 0$.*

Proof. If $y = 0$, we have $E(x,0) = ax^2$, so $E(x,0) > 0$ for $x \neq 0$ if and only if $a > 0$. If $y \neq 0$, we have

$$E(x,y) = y^2\left[a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c\right];$$

and when $a > 0$ the bracketed polynomial in x/y (which is positive for large x/y) is positive for all x/y if and only if $b^2 - 4ac < 0$. This proves the first part of the theorem, and the second part follows at once by considering the function $-E(x,y)$.

PROBLEMS

1. Determine whether each of the following functions is positive definite, negative definite, or neither:
(a) $x^2 - xy - y^2$;
(b) $2x^2 - 3xy + 3y^2$;
(c) $-2x^2 + 3xy - y^2$;
(d) $-x^2 - 4xy - 5y^2$.

2. Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

3. Show that (0,0) is an asymptotically stable critical point for each of the following systems:

(a) $\begin{cases} \dfrac{dx}{dt} = -3x^3 - y \\ \dfrac{dy}{dt} = x^5 - 2y^3; \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = -2x + xy^3 \\ \dfrac{dy}{dt} = -x^2y^2 - y^3. \end{cases}$

4. Prove that the critical point (0,0) of the system (1) is unstable if there exists a function $E(x,y)$ with the following properties:
(a) $E(x,y)$ is continuous and has continuous first partial derivatives in some region containing the origin;
(b) $E(0,0) = 0$;
(c) every circle centered on (0,0) contains at least one point where $E(x,y)$ is positive;
(d) $(\partial E/\partial x)F + (\partial E/\partial y)G$ is positive definite.

5. Show that (0,0) is an unstable critical point for the system

$$\begin{cases} \dfrac{dx}{dt} = 2xy + x^3 \\ \dfrac{dy}{dt} = -x^2 + y^5. \end{cases}$$

6. Assume that $f(x)$ is a function such that $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$ [that is, $f(x) > 0$ when $x > 0$ and $f(x) < 0$ when $x < 0$].
(a) Show that

$$E(x,y) = \frac{1}{2}y^2 + \int_0^x f(x)\,dx$$

is positive definite.
(b) Show that the equation

$$\frac{d^2x}{dt^2} + f(x) = 0$$

has $x = 0$, $y = dx/dt = 0$ as a stable critical point.
(c) If $g(x) \geq 0$ in some neighborhood of the origin, show that the equation

$$\frac{d^2x}{dt^2} + g(x)\frac{dx}{dt} + f(x) = 0$$

has $x = 0$, $y = dx/dt = 0$ as a stable critical point.

62 SIMPLE CRITICAL POINTS OF NONLINEAR SYSTEMS

Consider an autonomous system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \tag{1}$$

with an isolated critical point at (0,0). If $F(x,y)$ and $G(x,y)$ can be expanded in power series in x and y, then (1) takes the form

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + c_1x^2 + d_1xy + e_1y^2 + \cdots \\ \dfrac{dy}{dt} = a_2x + b_2y + c_2x^2 + d_2xy + e_2y^2 + \cdots. \end{cases} \tag{2}$$

When $|x|$ and $|y|$ are small—that is, when (x,y) is close to the origin—the terms of second degree and higher are very small. It is therefore natural to discard these nonlinear terms and conjecture that the qualitative behavior of the paths of (2) near the critical point (0,0) is similar to that of the paths of the related linear system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y. \end{cases} \tag{3}$$

We shall see that in general this is actually the case. The process of replacing (2) by the linear system (3) is usually called *linearization.*

More generally, we shall consider systems of the form

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + f(x,y) \\ \dfrac{dy}{dt} = a_2x + b_2y + g(x,y). \end{cases} \tag{4}$$

It will be assumed that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \tag{5}$$

so that the related linear system (3) has (0,0) as an isolated critical point; that $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives for all (x,y); and that as $(x,y) \to (0,0)$ we have

$$\lim \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0 \qquad \text{and} \qquad \lim \frac{g(x,y)}{\sqrt{x^2+y^2}} = 0. \tag{6}$$

Observe that conditions (6) imply that $f(0,0) = 0$ and $g(0,0) = 0$, so (0,0) is a critical point of (4); also, it is not difficult to prove that this critical point is isolated (see Problem 1). With the restrictions listed above, (0,0) is said to be a *simple critical point* of the system (4).

Example 1. In the case of the system

$$\begin{cases} \dfrac{dx}{dt} = -2x + 3y + xy \\ \dfrac{dy}{dt} = -x + y - 2xy^2 \end{cases} \tag{7}$$

we have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} = 1 \neq 0,$$

so (5) is satisfied. Furthermore, by using polar coordinates we see that

$$\frac{|f(x,y)|}{\sqrt{x^2+y^2}} = \frac{|r^2 \sin\theta \cos\theta|}{r} \le r$$

and

$$\frac{|g(x,y)|}{\sqrt{x^2+y^2}} = \frac{|2r^3 \sin^2\theta \cos\theta|}{r} \le 2r^2,$$

so $f(x,y)/r$ and $g(x,y)/r \to 0$ as $(x,y) \to (0,0)$ (or as $r \to 0$). This shows that conditions (6) are also satisfied, so (0,0) is a simple critical point of the system (7).

The main facts about the nature of simple critical points are given in the following theorem of Poincaré, which we state without proof.[9]

Theorem A. *Let* (0,0) *be a simple critical point of the nonlinear system* (4), *and consider the related linear system* (3). *If the critical point* (0,0) *of* (3) *falls under any one of the three major cases described in Section* 60, *then the critical point* (0,0) *of* (4) *is of the same type.*

As an illustration, we examine the nonlinear system (7) of Example 1, whose related linear system is

$$\begin{cases} \dfrac{dx}{dt} = -2x + 3y \\ \dfrac{dy}{dt} = -x + y. \end{cases} \tag{8}$$

The auxiliary equation of (8) is $m^2 + m + 1 = 0$, with roots

$$m_1, m_2 = \frac{-1 \pm \sqrt{3}\,i}{2}.$$

[9] Detailed treatments can be found in W. Hurewicz, *Lectures on Ordinary Differential Equations*, pp. 86–98, MIT, Cambridge, Mass., 1958; L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, pp. 157–163, Academic Press, New York, 1963; or F. G. Tricomi, *Differential Equations*, pp. 53–72, Blackie, Glasgow, 1961.

Since these roots are conjugate complex but not pure imaginary, we have Case C and the critical point (0,0) of the linear system (8) is a spiral. By Theorem A, the critical point (0,0) of the nonlinear system (7) is also a spiral.

It should be understood that while the type of the critical point (0,0) is the same for (4) as it is for (3) in the cases covered by the theorem, the actual appearance of the paths may be somewhat different. For example, Fig. 76 shows a typical saddle point for a linear system, whereas Fig. 83 suggests how a nonlinear saddle point might look. A certain amount of distortion is clearly present in the latter, but nevertheless the qualitative features of the two configurations are the same.

It is natural to wonder about the two borderline cases, which are not mentioned in Theorem A. The facts are these: if the related linear system (3) has a borderline node at the origin (Case D), then the nonlinear system (4) can have either a node or a spiral; and if (3) has a center at the origin (Case E), then (4) can have either a center or a spiral. For example, (0,0) is a critical point for each of the nonlinear systems

$$\begin{cases} \dfrac{dx}{dt} = -y - x^2 \\ \dfrac{dy}{dt} = x \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dx}{dt} = -y - x^3 \\ \dfrac{dy}{dt} = x. \end{cases} \tag{9}$$

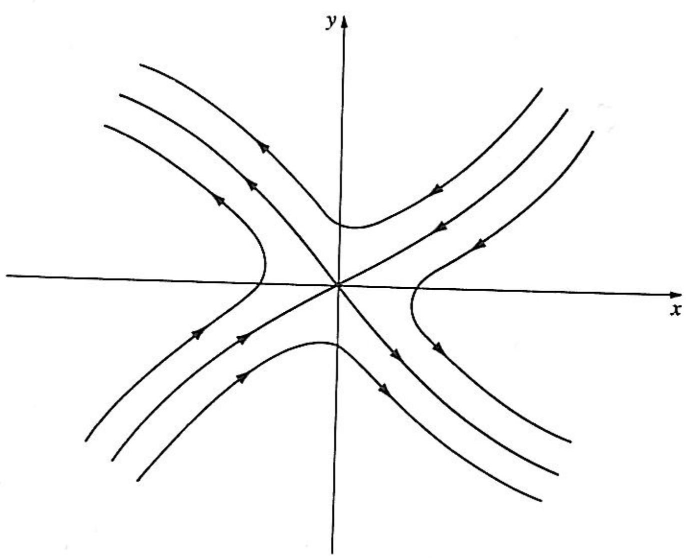


FIGURE 83

In each case the related linear system is

$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = x. \end{cases} \tag{10}$$

It is easy to see that (0,0) is a center for (10). However, it can be shown that while (0,0) is a center for the first system of (9), it is a spiral for the second.[10]

We have already encountered a considerable variety of configurations at critical points of linear systems, and the above remarks show that no new phenomena appear at simple critical points of nonlinear systems. What about critical points that are not simple? The possibilities here can best be appreciated by examining a nonlinear system of the form (2). If the linear terms in (2) do not determine the pattern of the paths near the origin, then we must consider the second degree terms; if these fail to determine the pattern, then the third degree terms must be taken into account, and so on. This suggests that in addition to the linear configurations, a great many others can arise, of infinite variety and staggering complexity. Several are shown in Fig. 84. It is perhaps surprising to realize that such involved patterns as these can occur in connection with systems of rather simple appearance. For example, the three figures in the upper row show the arrangement of the paths of

$$\begin{cases} \dfrac{dx}{dt} = 2xy \\ \dfrac{dy}{dt} = y^2 - x^2, \end{cases} \quad \begin{cases} \dfrac{dx}{dt} = x^3 - 2xy^2 \\ \dfrac{dy}{dt} = 2x^2y - y^3, \end{cases} \quad \begin{cases} \dfrac{dx}{dt} = x - 4y\sqrt{|xy|} \\ \dfrac{dy}{dt} = -y + 4x\sqrt{|xy|}. \end{cases}$$

In the first case, this can be seen at once by looking at Fig. 3 and equation 3-(8).

We now discuss the question of stability for a simple critical point. The main result here is due to Liapunov: if (3) is asymptotically stable at the origin, then (4) is also. We state this formally as follows.

Theorem B. *Let* (0,0) *be a simple critical point of the nonlinear system* (4), *and consider the related linear system* (3). *If the critical point* (0,0) *of* (3) *is asymptotically stable, then the critical point* (0,0) *of* (4) *is also asymptotically stable.*

[10] See Hurewicz, *op. cit.*, p. 99.

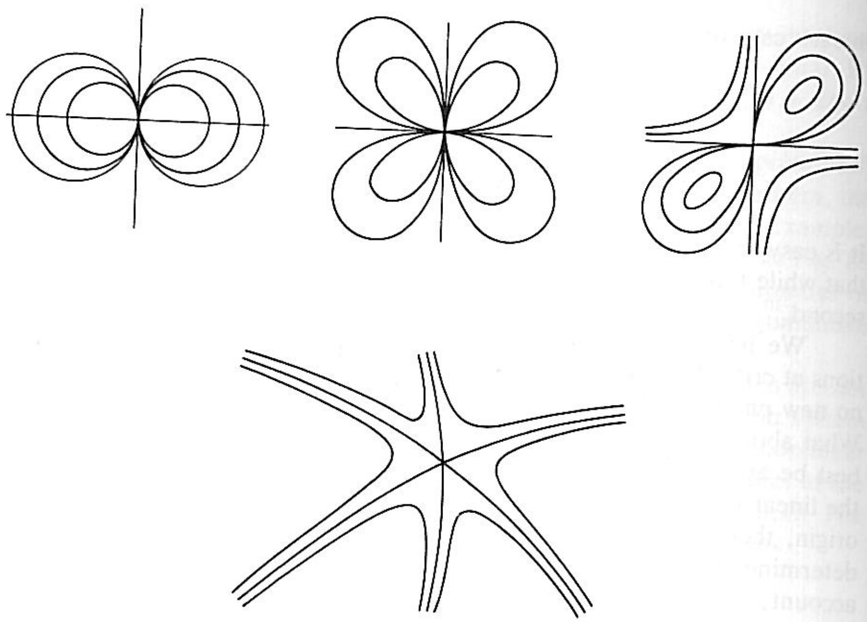


FIGURE 84

***Proof*.** By Theorem 61-A, it suffices to construct a suitable Liapunov function for the system (4), and this is what we do.

Theorem 60–B tells us that the coefficients of the linear system (3) satisfy the conditions

$$p = -(a_1 + b_2) > 0 \quad \text{and} \quad q = a_1b_2 - a_2b_1 > 0. \tag{11}$$

Now define

$$E(x,y) = \frac{1}{2}(ax^2 + 2bxy + cy^2)$$

by putting

$$a = \frac{a_2^2 + b_2^2 + (a_1b_2 - a_2b_1)}{D},$$

$$b = -\frac{a_1a_2 + b_1b_2}{D},$$

and

$$c = \frac{a_1^2 + b_1^2 + (a_1b_2 - a_2b_1)}{D},$$

where

$$D = pq = -(a_1 + b_2)(a_1b_2 - a_2b_1).$$

By (11), we see that $D > 0$ and $a > 0$. Also, an easy calculation shows that

$$\begin{aligned} D^2(ac - b^2) &= (a_2^2 + b_2^2)(a_1^2 + b_1^2) \\ &\quad + (a_2^2 + b_2^2 + a_1^2 + b_1^2)(a_1b_2 - a_2b_1) \\ &\quad + (a_1b_2 - a_2b_1)^2 - (a_1a_2 + b_1b_2)^2 \\ &= (a_2^2 + b_2^2 + a_1^2 + b_1^2)(a_1b_2 - a_2b_1) \\ &\quad + 2(a_1b_2 - a_2b_1)^2 \\ &> 0, \end{aligned}$$

so $b^2 - ac < 0$. Thus, by Theorem 61-B, we know that the function $E(x,y)$ is positive definite. Furthermore, another calculation (whose details we leave to the reader) yields

$$\frac{\partial E}{\partial x}(a_1x + b_1y) + \frac{\partial E}{\partial y}(a_2x + b_2y) = -(x^2 + y^2). \tag{12}$$

This function is clearly negative definite, so $E(x,y)$ is a Liapunov function for the linear system (3).[11]

We next prove that $E(x,y)$ is also a Liapunov function for the nonlinear system (4). If F and G are defined by

$$F(x,y) = a_1x + b_1y + f(x,y)$$

and

$$G(x,y) = a_2x + b_2y + g(x,y),$$

then since E is known to be positive definite, it suffices to show that

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G \tag{13}$$

is negative definite. If we use (12), then (13) becomes

$$-(x^2 + y^2) + (ax + by)f(x,y) + (bx + cy)g(x,y);$$

and by introducing polar coordinates we can write this as

$$-r^2 + r[(a\cos\theta + b\sin\theta)f(x,y) + (b\cos\theta + c\sin\theta)g(x,y)].$$

Denote the largest of the numbers $|a|$, $|b|$, $|c|$ by K. Our assumption (6) now implies that

$$|f(x,y)| < \frac{r}{6K} \quad \text{and} \quad |g(x,y)| < \frac{r}{6K}$$

for all sufficiently small $r > 0$, so

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G < -r^2 + \frac{4Kr^2}{6K} = -\frac{r^2}{3} < 0$$

[11] The reason for the definitions of a, b, and c can now be understood: we want (12) to be true.

for these r's. Thus $E(x,y)$ is a positive definite function with the property that (13) is negative definite. Theorem 61-A now implies that (0,0) is an asymptotically stable critical point of (4), and the proof is complete.

To illustrate this theorem, we again consider the nonlinear system (7) of Example 1, whose related linear system is (8). For (8) we have $p = 1 > 0$ and $q = 1 > 0$, so the critical point (0,0) is asymptotically stable, both for the linear system (8) and for the nonlinear system (7).

Example 2. We know from Section 58 that the equation of motion for the damped vibrations of a pendulum is

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{g}{a}\sin x = 0,$$

where c is a positive constant. The equivalent nonlinear system is

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{g}{a}\sin x - \dfrac{c}{m}y. \end{cases} \tag{14}$$

Let us now write (14) in the form

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{g}{a}x - \dfrac{c}{m}y + \dfrac{g}{a}(x - \sin x). \end{cases} \tag{15}$$

It is easy to see that

$$\frac{x - \sin x}{\sqrt{x^2 + y^2}} \to 0$$

as $(x,y) \to (0,0)$, for if $x \neq 0$, we have

$$\frac{|x - \sin x|}{\sqrt{x^2 + y^2}} \leq \frac{|x - \sin x|}{|x|} = \left|1 - \frac{\sin x}{x}\right| \to 0;$$

and since (0,0) is evidently an isolated critical point of the related linear system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{g}{a}x - \dfrac{c}{m}y, \end{cases} \tag{16}$$

it follows that (0,0) is a simple critical point of (15). Inspection shows ($p = c/m > 0$ and $q = g/a > 0$) that (0,0) is an asymptotically stable critical point of (16), so by Theorem B it is also an asymptotically stable critical point of (15). This reflects the obvious physical fact that if the pendulum is slightly disturbed, then the resulting motion will die out with the passage of time.

PROBLEMS

1. Prove that if (0,0) is a simple critical point of (4), then it is necessarily isolated. *Hint*: Write conditions (6) in the form $f(x,y)/r = \epsilon_1 \to 0$ and $g(x,y)/r = \epsilon_2 \to 0$, and in the light of (5) use polar coordinates to deduce a contradiction from the assumption that the right sides of (4) both vanish at points arbitrarily close to the origin but different from it.
2. Sketch the family of curves whose polar equation is $r = a \sin 2\theta$ (see Fig. 84), and express the differential equation of this family in the form $dy/dx = G(x,y)/F(x,y)$.
3. If (0,0) is a simple critical point of (4) and $q = a_1b_2 - a_2b_1 < 0$, then Theorem A implies that (0,0) is a saddle point of (4) and is therefore unstable. Prove that if $p = -(a_1 + b_2) < 0$ and $q = a_1b_2 - a_2b_1 > 0$, then (0,0) is an unstable critical point of (4). *Hint*: Adapt the proof of Theorem B to show that there exists a positive definite function $E(x,y)$ such that

$$\frac{\partial E}{\partial x}(a_1x + b_1y) + \frac{\partial E}{\partial y}(a_2x + b_2y) = x^2 + y^2,$$

and apply Problem 61-4. (Observe that these facts together with Theorem B demonstrate that all the information in Fig. 80 about asymptotic stability and instability carries over directly to nonlinear systems with simple critical points from their related linear systems.)
4. Show that (0,0) is an asymptotically stable critical point of

$$\begin{cases} \dfrac{dx}{dt} = -y - x^3 \\ \dfrac{dy}{dt} = x - y^3, \end{cases}$$

but is an unstable critical point of

$$\begin{cases} \dfrac{dx}{dt} = -y + x^3 \\ \dfrac{dy}{dt} = x + y^3. \end{cases}$$

How are these facts related to the parenthetical remark in Problem 3?
5. Verify that (0,0) is a simple critical point for each of the following systems, and determine its nature and stability properties:

(a) $\begin{cases} \dfrac{dx}{dt} = x + y - 2xy \\ \dfrac{dy}{dt} = -2x + y + 3y^2; \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = -x - y - 3x^2y \\ \dfrac{dy}{dt} = -2x - 4y + y \sin x. \end{cases}$

6. The van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$